MEASURABLE CARDINALS AND THE CONTINUUM HYPOTHESIS

BY

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ABSTRACT

Let ZFM be the set theory ZF together with an axiom which asserts the existence of a measurable cardinal. It is shown that if ZFM is consistent then ZFM is consistent with every sentence ϕ whose consistency is proved by Cohen's forcing method with a set of conditions of cardinality $< \kappa$. In particular, if ZFM is consistent then it is consistent with the continuum hypothesis and with its negation.

1. **Introduction.** The two major questions which are left unanswered by the Zermelo-Fraenkel axiom system, ZF,[2] of set theory are, roughly: How "many" subsets does a given set have, and how "big" can cardinal numbers get. Various axioms have been considered, which answer, to some extent, these questions. This suggests the following problem: How do suggested answers to one of these questions affect the answer to the other one.

It turned out that the strong axioms of infinity traditionally considered did not have any effect on the number of subsets of a given set; these axioms did not even settle the question of whether a given set has a non-constructible subset. (These results are due to Cohen and Gödel.) However, when one considers the axiom which asserts the existence of a measurable cardinal, the situation changes radically. It was first shown by Scott [9] that this axiom entails the existence of non-constructible sets; later work by Rowbottom, Gaifman, Silver and Solovay showed that from this axiom one can draw far-reaching conclusions concerning the existence of non-constructible sets of integers. Thus one can show, assuming there are measurable cardinals, that there are many non-constructible subsets of ω. Can one show, using the same assumption, the existence of more than $\aleph_1$ subsets of ω, and thereby refute the continuum hypothesis?

Unfortunately the answer is no. We shall prove, in this paper, the following theorem:

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[2] Throughout this paper, we consider the axiom of choice to be an axiom of ZF.

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THEOREM 1. *Suppose that ZF +* "there is a measurable cardinal" *is a consistent theory. Then both the continuum hypothesis and its negation can be consistently added to this theory.*

Our principal tool in proving Theorem 1 is Cohen's forcing method [1]. In order to verify that there are measurable cardinals in the Cohen extension, we prove a technical theorem, Theorem 3 below, which states, roughly, that if κ is a measurable cardinal, and the set of conditions has power less than κ, then κ remains measurable in the Cohen extension.

We shall prove a strengthened version of Theorem 1, which considers the value of $2^{\aleph_\alpha}$ for a suitable initial segment of the α's. Before stating our result, we recall a result of Easton [2]. Given a function G such that one can prove in ZF

(i) $\alpha < \beta \rightarrow G(\alpha) \leqq G(\beta)$

and

(ii) $\aleph_{G(\alpha)}$ is not cofinal with cardinals $\leqq \aleph_\alpha$

then if ZF is consistent, it remains consistent when we add the axiom

$$\forall\alpha\,(\aleph_\alpha \textit{ is regular} \rightarrow 2^{\aleph_\alpha} = \aleph_{G(\alpha)}),$$

provided that the definition of G is of a particularly simple form. The rigorous requirement on G is that G should be absolute with respect to the Cohen-extension used in [2], but for our purposes it is enough that $G(\alpha)$ can be given by a term which involves only functions on the ordinals which have definitions refering only to constructible sets, such as the constants 0, 1 and the binary operations $x + y$ and $x \cdot y$, together with the (possibly non-constructible) unary operation ω_x. For example, we can take $G(\alpha) = \omega_{\alpha\cdot\alpha} + \omega_{\alpha\cdot 2+7}$. These restrictions on G are essential; if, for example, we are allowed to refer to the operation of cardinal exponentiation in the definition of G, we could choose $G(\alpha) = 2^{2^{\aleph_\alpha}}$; for this G, $2^{\aleph_\alpha} = \aleph_{G(\alpha)}$ is obviously contradictory.

We denote by ZFM the theory obtained from ZF by adding to it the statement that there exists a measurable cardinal. The following theorem gives a partial description of the possible behaviour, relative to ZFM, of the function $2^{\aleph_\alpha}$ (for regular $\aleph_\alpha$'s).

THEOREM 2. *Let G_1 and G_2 be functions on the ordinals for which one can prove in ZFM (i) and (ii) above; let the functions G_1, G_2 and a particular ordinal Γ be given by definitions which are absolute as required above; let the following facts be provable in ZFM: (For brevity, we denote by κ_0 the least measurable cardinal.)*

(iii) $\Gamma < \kappa_0$.

(iv) $\forall\alpha(\alpha < \Gamma \rightarrow G_1(\alpha) < \Gamma)$.

(v) $\forall\alpha(\Gamma \leqq \alpha \leqq \kappa_0 \rightarrow G_1(\alpha) = \alpha + 1)$.

Define a function $G: On \to On$ *by*

$$G(\alpha) = G_1(\alpha) \text{ for } \alpha \leqq \kappa_0,$$

$$G(\alpha) = G_2(\alpha) \text{ for } \alpha > \kappa_0.$$

(It is clear that G satisfies (i) and (ii) above.) Then if ZFM is consistent, it is compatible with the axiom

$$\forall\alpha(\aleph_\alpha \textit{ regular} \to 2^{\aleph_\alpha} = \aleph_{G(\alpha)}).$$

The proof of Theorem 2 uses a recent result of Silver [11] which states that the generalized continuum hypothesis is relatively consistent with ZFM. Theorem 2 allows us to manipulate the values of $2^{\aleph_\alpha}$ (for $\aleph_\alpha$ regular) in the ranges $\alpha < \Gamma$ and $\alpha > \kappa_0$ but gives no information (beyond that given by Silver's result) in the range $\Gamma \leqq \alpha \leqq \kappa_0$. Thus the following questions are open:

1) Is $2^{\aleph_{\kappa_0}} = \aleph_{\kappa_0+2}$ consistent with ZFM?

2) Is $\forall\alpha(\alpha < \kappa_0$ *and* $\aleph_\alpha$ *regular* $\to 2^{\aleph_\alpha} = \aleph_{\alpha+2})$ consistent with ZFM?

3) More generally, let $G(\alpha)$ be given by an absolute definition of the sort described preceding Theorem 2, and suppose that in ZFM, one can prove (i) and (ii) above. (For example, let $G(\alpha) = \alpha + 2$.) Is $\forall\alpha(\aleph_\alpha$ *regular* $\to 2^{\aleph_\alpha} = \aleph_{G(\alpha)})$ compatible with ZFM?

(It seems likely that the answer to each of these questions is "Yes".)

Our formulation of 3) does not allow for the notion of a measurable cardinal to appear in the definition of G. If we do allow this notion, then requirements (i) and (ii) are definitely not sufficient. Of course we have to rule out anything like $G(0) = \kappa_0$, but there are more subtle requirements relating $G(\kappa_0)$ to the values of $G(\alpha)$ for $\alpha < \kappa_0$. For example, it is shown in Hanf-Scott [3], that if $2^{\aleph_\alpha} \leqq \aleph_{\alpha+2}$, for all regular $\aleph_\alpha$ less than κ_0, then $2^{\aleph_{\kappa_0}} \leqq \aleph_{\kappa_0+2}$.

The remainder of this paper is organized as follows: in §2, we write down axioms for a forcing relation, which allow us to consider simultaneously the case of Cohen extensions and that of Boolean valued models. (Cf. [10]. That we can treat these two cases together is not surprising in view of the close relations between them established in [10].) In §3, we reduce the proofs of Theorems 1 and 2 to the proof of a certain technical result (Theorem 3 below). In §4, we give a proof of Theorem 3 together with a partial converse. Theorem 3 allows one to extend many other results from the case of ZF to that of ZFM: For example, Souslin's Hypothesis and its negation, Kurepa's Conjecture and its negation ([12]), and the proposition that all projective sets are Lebesgue measurable ([14]) are all relatively consistent with ZFM.

2. **Axioms for a forcing relation.** It is well-known to people working with Cohen's method that the use of countable models is not essential for consistency results. All that is really needed is a "forcing relation" having suitable properties.

In this section, we write down a set of axioms for a forcing relation, sufficient to carry out the proofs in this paper. These axioms will be simultaneously applicable to the forcing relations introduced by Cohen and to the Boolean valued models introduced by Scott and Solovay [10]. In reading the proofs in later sections, the reader may prefer to check our arguments using his intuitive knowledge of forcing, rather than the axioms.

By a class, in this paper, we always mean a subcollection of the universe of sets of the form $\{x \mid \phi(x,t)\}$. Here ϕ is a formula of ZF with two free variables and the set t serves as a parameter. For example, each set t is a class since $t = \{x \mid x \in t\}$. By a Cohen extension of set theory we mean the following. We are given a class C, the members of which we call *conditions*; a reflexive partial ordering $\preccurlyeq$ on C with a minimal member which we denote by 0; a class T, the members of which we call *terms*; finally, there is a one-one mapping $\{x \to \mathbf{x}\}$ of the class of all sets into T.

We let $\mathscr{L}$ be a first order language which includes the language of set theory (and which possibly contains extra relation symbols and special quantifiers). For every formula $\phi(x_1, \cdots, x_n)$ of $\mathscr{L}$, with no free variables other than $x_1, \cdots, x_n$, we suppose given a subclass $\Vdash_\phi$ of $C \times T^n$. We write

$$p \Vdash \phi(t_1, \cdots, t_n) \tag{1}$$

for $\langle p, t_1, \cdots, t_n \rangle \in \Vdash_\phi$. We read (1) as "$p$ forces $\phi(t_1, \cdots, t_n)$" (where by 'forcing' we intend what is frequently referred to as 'weak forcing', i.e. 'forcing of the double negation'). This completes our description of the "data" of the concep "Cohen extension".

When one starts with a countable model M of ZF and extends it, by Cohen's method, to another model N of ZF the sentences ϕ for which $0 \Vdash \phi$ is true in M are true in N; moreover if ϕ is a sentence of set theory (which does not contain the additional symbols of $\mathscr{L}$) and ϕ is true in N, then in many instances of the use of Cohen's method, $0 \Vdash \phi$ is true in M. Each set in N is denoted by some term $t \in T$. In particular, if $x \in M \subseteq N$, x is denoted by the term $\mathbf{x}$. We shall assume that $\mathscr{L}$ contains a unary predicate $S(x)$, which reads 'x *is standard*' and which is satisfied in N exactly by the members of M.

Even though in the following we shall not be dealing with countable models M and N (but talking only of forcing relations) we shall say that 'ϕ is true in the extension' for $0 \Vdash \phi$. We shall say 'ϕ is true in the ground model' if ϕ is true. If ϕ is a formula of the language of set theory, let ϕ^S be the formula obtained by relativizing the quantifiers with S; i.e.

$$[\forall x \psi(x)]^S \equiv_{df} [\forall x(S(x) \to \psi^S(x))];$$

$$[\exists x \psi(x)]^S \equiv_{df} [\exists x(S(x) \land \psi^S(x))];$$

relativization commutes with the other logical connectives, and is the identity on atomic formulas. Then $\phi(x_1, \cdots, x_n)$ holds in the ground model iff $\phi^S(\mathbf{x}_1, \cdots, \mathbf{x}_n)$ is true in the extension. Thus the predicate S allows us to talk about the "ground model" in $\mathscr{L}$.

We use the letters p, q (possibly with primes attached) to denote members of C. The classes $\Vdash_\phi$ are required to satisfy the following axioms:

(a) $p \Vdash \neg\phi(t_1, \cdots, t_n)$ if and only if for no $q \geqslant p$ does $q \Vdash \phi(t_1, \cdots, t_n)$.

(b) $p \Vdash \phi(t_1, \cdots, t_m) \vee \psi(t_1, \cdots, t_m)$ if and only if for every $q \geqslant p$ there is a $q' \geqslant q$ such that $q' \Vdash \phi(t_1, \cdots, t_m)$ or $q' \Vdash \psi(t_1, \cdots, t_m)$; $p \Vdash \exists x\, \phi(t_1, \cdots, t_n, x)$ if and only if for every $q \geqslant p$ there is a $q' \geqslant q$ and a $t \in T$ such that $q' \Vdash \phi(t_1, \cdots, t_n, t)$.

(c) $p \Vdash \phi(t_1, \cdots, t_n) \wedge \psi(t_1, \cdots, t_n)$ if and only if $p \Vdash \phi(t_1, \cdots, t_n)$ and $p \Vdash \psi(t_1, \cdots, t_n)$; $p \Vdash \forall x \phi(t_1, \cdots, t_n, x)$ if and only if for every $t \in T$, $p \Vdash \phi(t_1, \cdots, t_n, t)$.

(d) $p \Vdash S(t)$ if and only if for every $q \geqslant p$ there is a $q' \geqslant q$ and a set x such that $q' \Vdash t = \mathbf{x}$. $p \Vdash t \in \mathbf{a}$ if and only if for every $q \geqslant p$ there is a $q' \geqslant q$ and a set $x \in a$ such that $q' \Vdash t = \mathbf{x}$.

(e) Let $\phi(x_1, \cdots, x_n)$ and $\psi(x_1, \cdots, x_k)$ be formulas of $\mathscr{L}$. Let $t_1, \cdots, t_n, s_1, \cdots, s_k$ be terms from T such that the sentences $\phi(t_1, \cdots, t_n)$ and $\psi(s_1, \cdots, s_k)$ coincide. Then $p \Vdash \phi(t_1, \cdots, t_n)$ if and only if $p \Vdash \psi(s_1, \cdots, s_k)$. (Example: let ϕ be "$x_1 = x_2$", ψ be "$x_1 = x_1$" and let $t_1 = t_2 = s_1$.).

(f) If $\phi(x_1, \cdots, x_n)$ is an axiom of logic or ZF then $p \Vdash \phi(t_1, \cdots, t_n)$. (In the case of the replacement schema, we include all the instances expressible in $\mathscr{L}$; i.e., we allow predicates other than $\in$ or $=$ to occur.) If $p \Vdash \phi(t_1, \cdots, t_n)$ and $p \Vdash \phi(t_1, \cdots, t_n) \rightarrow \psi(t_1, \cdots, t_n)$ then $p \Vdash \psi(t_1, \cdots, t_n)$. Hence if $\phi(x_1, \cdots, x_n)$ is a theorem of ZF then $p \Vdash \phi(t_1, \cdots, t_n)$.

REMARKS. We can now give a sense to $p \Vdash \phi(t_1, \cdots, t_n)$ even if ϕ contains symbols which do not belong to the primitive language of ZF but are defined symbols of ZF (or, for that matter, if ϕ is formulated in English); in this case we mean by "$p \Vdash \phi(t_1, \cdots, t_n)$" that $p \Vdash \phi'(t_1, \cdots, t_n)$ where ϕ' is a formula which uses only primitive symbols of ZF and which is equivalent to ϕ; from (f) it follows that the truth or falsity of $p \Vdash \phi'(t_1, \cdots, t_n)$ does not depend on the particular choice of ϕ'.

Notice also that if $\phi(t_1, \cdots, t_n)$ and $\psi(t_1, \cdots, t_n)$ contradict one another in ZF then we cannot have $p \Vdash \phi(t_1, \cdots, t_n)$ together with $p \Vdash \psi(t_1, \cdots, t_n)$. In fact, from $p \Vdash \phi(t_1, \cdots, t_n)$ and $p \Vdash \phi(t_1, \cdots, t_n) \rightarrow \neg\psi(t_1, \cdots, t_n)$ (true by (*f*)) we get $p \Vdash \neg\psi(t_1, \cdots, t_n)$, and this contradicts $p \Vdash \psi(t_1, \cdots, t_n)$, by (a).

The following assumption is not needed for the proof of Theorem 3, but it holds in all the applications:

(g) $0 \Vdash \forall x(x \textit{ is an ordinal} \rightarrow S(x))$.

From (a)–(f) we obtain the following consequences.

(h) If $p \Vdash \phi(t_1, \cdots, t_n)$ and $q \geqslant p$, then also $q \Vdash \phi(t_1, \cdots, t_n)$ (this follows by applying (a) and (f) to $\neg\neg\phi(t_1, \cdots, t_n)$).

(i) $p \Vdash \phi(t_1, \cdots, t_n) \rightarrow \psi(t_1, \cdots, t_n)$ if and only if for every $q \geqslant p$ such that

$q \Vdash \phi(t_1, \cdots, t_n)$ there is $q' \succcurlyeq q$ such that $q' \Vdash \psi(t_1, \cdots, t_n)$. (This follows from (a), (b), (f) and (h).)

(j) $p \Vdash \forall x(S(x) \rightarrow \phi(t_1, \cdots, t_n, x))$ if and only if for every x, $p \Vdash \phi(t_1, \cdots, t_n, \mathbf{x})$; $p \Vdash \forall x(x \in \mathbf{a} \rightarrow \phi(t_1, \cdots, t_n, x))$ if and only if for every $x \in a$, $p \Vdash \phi(t_1, \cdots, t_n, \mathbf{x})$. (This follows from (d), (c), (f), (h) and (i).) There is an analogous fact concerning existential quantification.

We say that a formula $\phi(x_1, \cdots, x_n)$ of the language of ZF with no free variable other than $x_1, \cdots, x_n$ is absolute with respect to the extension if for all $x_1, \cdots, x_n$

$$0 \Vdash \phi(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n) \text{ if } \phi(x_1, \cdots, x_n)$$

and

$$0 \Vdash \neg \phi(\mathbf{x}_1, \cdots, \mathbf{x}_n) \text{ if } \neg \phi(x_1, \cdots, x_n).$$

Using (*d*), as well as (c), (f), (h), (i) and (j), one can show the following formulas are absolute: $x \in y$, $x = y$. (The proof is a simultaneous transfinite induction on $\max(\| x \|, \| y \|)$ where $\| x \|$ is the rank of the set x.) If $\phi(x_1, \cdots, x_n)$ and $\psi(x_1, \cdots, x_n)$ are absolute, so is $\neg \phi(x_1, \cdots, x_n)$ and $\phi(x_1, \cdots, x_n) \wedge \psi(x_1, \cdots, x_n)$. Moreover it follows from (j) that

$$(\exists x_1 \in y)\phi(x_1, \cdots, x_n) \text{ and } (\forall x_1 \in y)\phi(x_1, \cdots, x_n)$$

are absolute if ϕ is. Using these remarks, it is easy to verify the following:

(k) The following formulas are absolute: $x \in y$, $x = y$, $x \subseteq y$, $x \cap y = 0$, $\{x\} = y$, $x = y \cup z$, $\bigcup x = y$, *f is a (one-one) function from x onto (into) y (into the power-set of y)*, $z = \bigcup_{x \in y} f(x)$, *$x$ is an ordinal.*

This completes our discussion of the consequences of (a)–(g). We now discuss how we will use the formalism discussed here in later sections. Suppose that we have theories T_1 and T_2 at least as strong as ZF and we wish to prove

$$\text{(2)} \qquad \operatorname{Con}(T_1) \rightarrow \operatorname{Con}(T_2).$$

($\operatorname{Con}(T_i)$ is the number theoretic sentence expressing the consistency of T_i.) Then within the theory T_1 we construct a Cohen extension by giving definitions of C, T, $\Vdash_\phi$, etc. (One only has to give $\Vdash_\phi$ for atomic formulas ϕ; $\Vdash_\phi$ for more complicated formulas is then determined by a)–g).) We do this in such a way that a)–g) are theorem schemes of T_1. Finally, we show that for each axiom θ of T_2, $0 \Vdash \theta$ is a theorem of T_1. From this, (2) follows easily.

In practice, it is not necessary to explicitly describe the class of terms T. Suppose for example, that our conceptual picture is that we get the Cohen extension by adjoining a class of ordinals A. Then to describe the Cohen extentension, $M[A]$, we have only to specify 1) The partially ordered class C of conditions and 2) The forcing relation on the atomic sentences $\lambda \in A$ (for λ an ordinal). The construction of T, of $\Vdash_\phi$ for arbitrary formulas ϕ, etc. can be carried out exactly as in [7]. The requirements (a)–(g) hold by standard arguments if C is a set; if C is a class, it will not in general be true that $0 \Vdash \phi$ for each axiom ϕ of ZF.

However, this will be true, by results of Easton, in the one case when we make use of a class of conditions.

It is absolutely vital for the applications in §3 that we allow $V \neq L$ to hold in the ground model since $V = L$ is incompatible with the existence of a measurable cardinal. The presentation of forcing in [7] meets this requirement.

Our axioms (a)–(g) are also satisfied by the Boolean algebraic-valued interpretation of ZF of Scott-Solovay [10]. One takes C to be the set of non-zero elements of a "dense" subalgebra of the Boolean algebra. (By "dense" we mean that for every nonzero element a of the Boolean algebra, there is a $p \in C$ with $0 < p \leqq a$.) We say $p \preccurlyeq q$ if $q \leqq p$; T is the class of all "sets of the model"; $\mathbf{x}$ is the "standard set corresponding to x", and $p \Vdash \phi(t_1, \cdots, t_n)$ means that $p \leqq$ the truth value of $\phi(t_1, \cdots, t_n)$. Both Cohen's method and the method of Boolean-valued models apply to cases when the axiom of choice fails. Our axioms for forcing still hold in this case if one now interprets ZF to be Zermelo-Frankel set theory *excluding* the axiom of choice.

3. **Proofs of Theorems 1 and 2.**

Let $P(\alpha)$ be a property of ordinals expressible in ZF.

DEFINITION. P is preserved under mild Cohen extensions if whenever κ is a cardinal such that $P(\kappa)$ holds and the class of conditions C *is a set of power less than* κ then $P(\boldsymbol{\kappa})$ holds in the Cohen extension.

DEFINITION: A cardinal κ is measurable if it is uncountable and the Boolean algebra $S(\kappa)$ of all subsets of κ has a non-principal κ-complete prime ideal. ("κ-complete" means closed under unions of power less than κ.)

Alternatively, an uncountable cardinal κ is measurable if there is a κ-additive measure $\mu: S(\kappa) \to \{0,1\}$ vanishing on points and giving κ the measure 1.

Our definition of measurable cardinal differs from that in Scott [9] in that he only requires μ to be countably additive. However the least measurable cardinal in the sense of [9] is measurable in our sense. If κ is measurable, κ is the κth strongly inaccessible cardinal. (All these results are proved in [5]. In [5], the measurable cardinals are referred to as "uncountable cardinals not in C_1.")

The following theorem is the key new idea of the present paper. It will be proved in §4.

THEOREM 3. *The property of measurability is preserved under mild Cohen extensions.*

We now prove Theorem 1. Theorem 1 will follow immediately from Theorem 3 and the following theorem.

THEOREM 4. *Let $P(\alpha)$ be a property of cardinals which is preserved under mild Cohen extensions. Suppose further that*

$$\vdash_{ZF} \forall\alpha(P(\alpha) \to \alpha \text{ is a strongly inaccessible cardinal}).$$

Then if ZF + "$\exists\alpha P(\alpha)$" is consistent, then so are the theories

$$\text{ZF} + \text{"}\exists\alpha P(\alpha)\text{"} + \text{"}2^{\aleph_0} = \aleph_1\text{"}$$

and

$$\text{ZF} + \text{"}\exists\alpha P(\alpha)\text{"} + \text{"}2^{\aleph_0} > \aleph_1\text{"}.$$

Proof. To get the second result we use the original Cohen extension of [1] to "adjoin" $\aleph_2$ generic subsets of ω. Then it is clear that $2^{\aleph_0} \geqq \aleph_2$ in this Cohen extension. Moreover, the set of conditions has power $\aleph_2$. From our assumptions on P it is clear that $\exists\alpha P(\alpha)$ holds in the Cohen extension if it holds in the ground model.

We now prove the first result: We are going to "adjoin" a map $F: \aleph_1 \to S(\omega)$ to the ground model. ($S(\omega)$ is the power set of ω.) A condition on F will simply specify the restriction of F to some countable ordinal, i.e.. a condition will be a function on a countable ordinal into $S(\omega)$. Thus it is clear that F maps $\aleph_1$ into the set of standard subsets of ω. An easy argument shows that F maps *onto* the standard subsets.

Now the conditions are closed under countable increasing unions. It follows (by Lemma 7 below) that every subset of ω in the Cohen extension is standard. Thus "$2^{\aleph_0} \leqq \aleph_1$" holds in the extension, so the continuum hypothesis holds in the extension.

Finally, the cardinality of the set of conditions is $2^{\aleph_0}$. Thus our assumptions on P imply that $\exists\alpha(P(\alpha))$ holds in the extension if it holds in the ground model. The proof of Theorem 4 is complete.

DISCUSSION: Nearly all of the large cardinal properties that have been considered to date are preserved under mild Cohen extensions. For example this is true of the properties: "Ramsey", "Strongly inaccessible", "Measurable", "Mahlo", "n-th-order indescribable", "Strongly compact", etc. For many of these properties, this fact is well-known. For "Strongly compact" this is a theorem of McAloon [8].

Let P be some property of cardinals. We can express that there are many cardinals with the property P by the axiom

$$\forall\alpha\, \exists\beta(\beta > \alpha \textit{ and } P(\beta)) \tag{3}$$

which asserts that there are arbitrarily large cardinals with the property P. An even stronger principle in this direction can be expressed using the concept of a normal function. A function F: $On \to On$ (On is the class of all ordinals) is *normal* if (1) $\alpha < \beta \to F(\alpha) < F(\beta)$, and (2) if λ is a limit ordinal, $F(\lambda) = \sup\{F(\beta): \beta < \lambda\}$). The following principle gives a scheme of first order axioms:

$$(\forall F)(F \textit{ normal} \to \exists\alpha(F(\alpha) = \alpha \textit{ and } P(\alpha))). \tag{4}$$

It is not difficult to show that (4) implies (3). (The function $\{\lambda \to \alpha + \lambda\}$ is normal.) Lévy has shown in [6] that (4) is equivalent to the following reflection schema:

$$\forall\alpha(\exists\beta > \alpha)(P(\beta) \wedge (\forall x_1, \cdots, x_n \in R(\beta))(\phi(x_1, \cdots, x_n) \leftrightarrow \phi^{R(\beta)}(x_1, \cdots, x_n))$$

(Here $R(\beta)$ is the set of all sets of rank less than β. $\phi^{R(\beta)}$ is obtained from ϕ by relativizing all quantifiers to $R(\beta)$.)

LEMMA 5. *Suppose that the class of conditions C for some Cohen extension is a set and that the property P of cardinals is preserved under mild Cohen extensions. Then if* (3) *(resp.* (4)*) holds in the ground model it holds in the extension.*

Proof. For (3) this is clear. Suppose then that (4) holds in the ground model. We show that it holds in the extension. Let F be a class of ordered pairs in the extension. That is, F is given by some formula $\phi(x, y, z)$ of $\mathscr{L}$ and a parameter t: $F = \{\langle x, y\rangle : \phi(x, y, t)\}$. Suppose also that some condition p forces "*F is a normal function*". It suffices to show that there is an ordinal λ such that $p \Vdash \phi(\boldsymbol{\lambda}, \boldsymbol{\lambda}, t)$ *and* $P(\boldsymbol{\lambda})$, since then we get, by the absoluteness of "*x is an ordinal*" (see (k)),

$$p \Vdash \boldsymbol{\lambda} \textit{ is an ordinal} \wedge \phi(\boldsymbol{\lambda}, \boldsymbol{\lambda}, t) \wedge P(\boldsymbol{\lambda}),$$

and (4) holds in the Cohen extension by the second part of (b) and by (i).

Suppose that $q \geqslant p$ and $q \Vdash (\phi(\boldsymbol{\alpha}, \boldsymbol{\beta}, t)$ *and* $\phi(\boldsymbol{\alpha}, \boldsymbol{\gamma}, t))$. Then since $p \Vdash F$ *is a function*, $q \Vdash \boldsymbol{\beta} = \boldsymbol{\gamma}$ and hence (by (k)) $\beta = \gamma$. It follows that $S_\alpha = \{\beta \mid (\exists q \geqslant p)\ q \Vdash \phi(\boldsymbol{\alpha}, \boldsymbol{\beta}, t)\}$ is a set. We define a function $G(\alpha)$ by transfinite induction: Let γ be the least infinite cardinal following the cardinal of C. We put:

$$\begin{aligned} G(0) &= \max(\operatorname{lub} S_0, \gamma); \\ G(\alpha+1) &= \max(\operatorname{lub} S_{\alpha+1}, G(\alpha)+1); \\ G(\lambda) &= \operatorname{lub}\{G(\alpha) : \alpha < \lambda\} \text{ for } \lambda \text{ a limit ordinal.} \end{aligned}$$

Then G is clearly normal. For α not a limit ordinal, we have, by construction,

(5) $$p \Vdash F(\boldsymbol{\alpha}) \leqq \boldsymbol{\beta}, \text{ where } \beta = G(\alpha).$$

(Cf. the definition of S_α.) Since $p \Vdash$ "*F is normal*", it is easy to see that (5) holds also when α is a limit ordinal.

Since (4) holds in the ground model, there is an ordinal λ such that $G(\lambda) = \lambda$ and $P(\lambda)$. Since $G(\lambda) \geqq G(0) \geqq \gamma >$ the cardinal of C, $P(\boldsymbol{\lambda})$ holds in the extension. Since $p \Vdash F$ *is increasing*, and since $G(\lambda) = \lambda$, $p \Vdash \boldsymbol{\lambda} \leqq F(\boldsymbol{\lambda}) \leqq \boldsymbol{\lambda}$. Thus $p \Vdash F(\boldsymbol{\lambda}) = \boldsymbol{\lambda}$ *and* $P(\boldsymbol{\lambda})$. q.e.d.

Using Theorem 3, Lemma 5 and the method of proof of Theorem 4, we can derive the following corollary.

THEOREM 6. *If the theory ZF +* "Every normal function has a measurable fixed point" *is consistent then both*

$$\text{“}2^{\aleph_0} = \aleph_1\text{”} \textit{ and } \text{“}2^{\aleph_0} \neq \aleph_1\text{”}$$

can be consistently added to it.

We now prove Theorem 2. We first invoke a recent theorem of Silver [11] which states that if ZFM is consistent, so is ZFM + GCH. (GCH is the generalized continuum hypothesis which states $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all ordinals α).

Our proof will be in two steps. In the first, we handle the values of $2^{\aleph_\alpha}$ for $\alpha \geqq \kappa_0 + 1$. In the second step we handle the values of $2^{\aleph_\alpha}$ for $\alpha \leqq \Gamma$.

For the first step, we consider a Cohen extension of the type considered in Easton [2]. For each regular cardinal $\aleph_\alpha$, greater than κ_0, we adjoin $\aleph_{G_2(\alpha)}$ subsets of $\aleph_\alpha$. The arguments of Easton show that if GCH holds in the ground model then the statements

1) $2^{\aleph} = \aleph_{\alpha+1}$ *for* $\alpha \leqq \boldsymbol{\kappa}_0$
2) $2^{\aleph_\alpha} = \aleph_{G_2(\alpha)}$ *for* $\alpha > \boldsymbol{\kappa}_0$

hold in the extension.

A priori, the ordinal $\boldsymbol{\kappa}_0$ need not be measurable in the extension. We show next, however, that the statement

3) $\boldsymbol{\kappa}_0$ *is the least measurable cardinal*

holds in the extension.

Our proof will use the following lemma.

LEMMA 7 (SOLOVAY [13]). *Let λ be an infinite cardinal. Suppose that the class of conditions has the property that every well-ordered increasing sequence of conditions of length at most λ has an upper bound in C. Then*

$$\forall x(x \cap \boldsymbol{\lambda} \textit{ is standard})$$

holds in the Cohen extension.

Proof. Let p be a condition, and t a term. We show that $\exists p' \succcurlyeq p$ such that $p' \Vdash t \cap \boldsymbol{\lambda}$ *is standard*. We first construct an increasing sequence $\{p_\alpha \mid 0 \leqq \alpha < \lambda\}$ of conditions such that 1) $p \preccurlyeq p_0$; 2) Either $p_\alpha \Vdash \boldsymbol{\alpha} \in t$ or $p_\alpha \Vdash \boldsymbol{\alpha} \notin t$. There is no difficulty in constructing such a sequence by transfinite induction in view of our assumption on C.Let p' be an upper bound for $\{p_\alpha \mid \alpha < \lambda\}$ in C. Let $a = \{\alpha \mid p' \Vdash \boldsymbol{\alpha} \in t\}$. We prove now $p' \Vdash t \cap \boldsymbol{\lambda} = \mathbf{a}$ as follows. By (f) it is enough to show

$$p' \Vdash \forall x(x \in \boldsymbol{\lambda} \to (x \in t \to x \in \mathbf{a})) \wedge \forall x(x \in \mathbf{a} \to x \in t \wedge x \in \boldsymbol{\lambda});$$

this can be easily shown by means of the second part of (j). In view of (d), Lemma 7 is proved.

Since the conditions we are considering are closed under unions of length at most κ_0, Lemma 7 implies that every subset of $\boldsymbol{\kappa}_0$ in the Cohen extension is standard. In exactly the same way we can also show that every subset of $\kappa_0 \times \kappa_0$ in the Cohen extension is standard, and therefore every function on a cardinal $\lambda < \kappa_0$ into $S(\kappa_0)$ in the Cohen extension is standard. It follows that if I is a κ_0-complete non-principal prime ideal in $S(\kappa_0)$, I is a κ_0-complete non-principal prime ideal on $S(\kappa_0)$ in the Cohen extension. Thus κ_0 remains measurable in the extension. Similarly, if λ is a cardinal $< \kappa_0$, every subset of λ is standard. It follows easily (since 2^λ is also less than κ_0), that λ is measurable iff λ is measurable in the Cohen extension. But κ_0 was the least measurable cardinal, therefore, κ_0 is the least measurable cardinal in the Cohen extension. We have proved the following lemma.

LEMMA 8. *Assume ZFM is consistent. Then ZFM remains consistent if we add the following axioms*:

1) $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ if $\alpha \leqq \kappa_0$,
2) $2^{\aleph_\alpha} = \aleph_{G_2(\alpha)}$ if $\alpha > \kappa_0$, and $\aleph_\alpha$ is regular.

(In 1), 2) κ_0 is *an abbreviation for "the least measurable cardinal"*.)

We shall now use another Cohen extension to deduce the consistency of the theory described in Theorem 2 from the consistency of the theory described in Lemma 8. The extension we use is again of the type considered in Easton [2]. For each regular cardinal $\aleph_\alpha < \Gamma$ (cf. the statement of Theorem 2 in §1) we adjoin $\aleph_{G_1(\alpha)}$ generic subsets of $\aleph_\alpha$. Then assuming that 1) and 2) hold in the ground model, it follows by arguments in [2] that the statement

$$\text{"}2^{\aleph_\alpha} = \aleph_{G_1(\alpha)} \textit{ for } \alpha \leqq \kappa_0, \textit{ and } \aleph_\alpha \textit{ regular};$$
$$2^{\aleph_\alpha} = \aleph_{G_2(\alpha)} \textit{ for } \alpha > \kappa_0, \textit{ and } \aleph_\alpha \textit{ regular."}$$

holds in the Cohen extension. To complete the proof it suffices to show that "κ_0 *is the least measurable cardinal*" holds in the Cohen extension.

Now an examination of Easton [2] shows that the cardinality of the set of conditions used is at most 2^Γ. Since ZFM $\vdash \Gamma < \kappa_0$, and κ_0 is strongly inaccessible, Theorem 3 implies that κ_0 is measurable in the Cohen extension. Suppose next that for some condition p and for some $\lambda < \kappa_0$,

$$p \Vdash \lambda \textit{ is measurable.} \tag{6}$$

We shall derive a contradiction. This will show that κ_0 is the least measurable cardinal in the Cohen extension and so complete the proof of Theorem 2.

From (6), it follows that $p \Vdash \lambda$ *is strongly inaccessible*. It follows from the definition of strong inaccessibility and from (k) that λ is strongly inaccessible. Moreover since the definition of Γ is "absolute" and ZFM $\vdash \Gamma$ *is less than the least measurable cardinal*, (6) implies that $p \Vdash \Gamma < \lambda$. It follows that $2^\Gamma < \lambda$. Since $\lambda < \kappa_0$, λ is not measurable. Thus (6) contradicts the following lemma which will be proved in §4 (and which is a partial converse to Theorem 3).

LEMMA 9. *The property of being a "non-measurable cardinal" is preserved under mild Cohen extensions.*

This completes our present discussion of the proof of Theorem 2.

REMARKS 1. R. Jensen [4] has announced the construction of a Cohen extension of an arbitrary model of ZF in which the generalized continuum hypothesis holds. It seems likely that, by using his techniques, one could prove a version of Theorem 2 for an arbitrary property, P, preserved under mild Cohen extensions. (The version would allow us, roughly speaking, to set $2^{\aleph_\alpha} = \aleph_{G_1(\alpha)}$ for regular $\aleph_\alpha < \Gamma$, where G_1 is as in Theorem 2, Γ is a cardinal with an "absolute" definition and

$$\vdash_{ZF} (\forall\alpha)(P(\alpha) \rightarrow \Gamma < \alpha \textit{ and } \alpha \textit{ is strongly inaccessible}).$$

2. We know of no systematic technique for proving "large cardinal" axioms compatible with GCH. In particular, the following questions are open, as far as we know:

1) Is ZF + GCH + "There is a compact cardinal" consistent (relative to ZF + "There is a compact cardinal")?

2) Same question as 1) but replace "There is a compact cardinal" by the axiom scheme "Every normal function has a measurable fixed point".

§4. **κ-complete prime ideals in Cohen extensions.** The following theorem is a sharpened version of Theorem 3.

THEOREM 10. *Let κ be a measurable cardinal and I a non-principal κ-complete prime ideal of the power set of κ, $S(\kappa)$. Suppose that the class, C, of conditions of some Cohen extension has power less than κ. Then it is true in the extension that I generates a non-principal $\boldsymbol{\kappa}$-complete prime ideal $J = \{x \subseteq \boldsymbol{\kappa}\colon (\exists y \in I)\, x \subseteq y\}$ of $S(\boldsymbol{\kappa})$.*

Proof. Let t be a term and p a condition. Let

$$T = \{\alpha < \kappa \colon (\exists q \succcurlyeq p)(q \Vdash \boldsymbol{\alpha} \in t)\}.$$

(Intuitively, T is the set of "possible" members of $t \cap \boldsymbol{\kappa}$.) Consider now the two possibilities: $T \in I$ and $T \notin I$:

Case 1. $T \in I$. It is now easy to prove $p \Vdash \forall x(x \in \boldsymbol{\kappa} \rightarrow (x \in t \rightarrow x \in \boldsymbol{T}))$, using the second part of (j), (i), and the absoluteness of $x \in y$ which is stated in (κ). Therefore we have, by the remark which follows (f), $p \Vdash t \cap \boldsymbol{\kappa} \subseteq \boldsymbol{T}$. If we let J be an abbreviation for "*the ideal generated by* $\boldsymbol{I}$ *in* $S(\boldsymbol{\kappa})$," then it follows that $p \Vdash (t \cap \boldsymbol{\kappa}) \in J$.

Case 2. $T \notin I$. Since I is κ-complete and the set of conditions, C, has power less than κ, it follows that for some $q \succcurlyeq p$, the set

$$W = \{\alpha < \boldsymbol{\kappa} \colon q \Vdash \boldsymbol{\alpha} \in t\}$$

is not in I. Using the second part of (j) we easily get $q \Vdash W \subseteq t$. Thus $q \Vdash (\boldsymbol{\kappa} - t) \subseteq V$, where $V = \kappa - W$. Since I is prime, $(\kappa - W) \in I$. It follows that $q \Vdash (\boldsymbol{\kappa} - t) \in J$.

Our discussion shows that for any term t and any condition p, some extension of p forces $(t \cap \boldsymbol{\kappa}) \in J$ or $(\boldsymbol{\kappa} - t) \in J$. It follows from (b) and (c) that

$$0 \Vdash J \textit{ is prime.}$$

Since I is non-principal, $0 \Vdash J$ *is non-principal*. To complete the proof, we show "*J is* $\boldsymbol{\kappa}$-*complete*" holds in the Cohen extension.

We suppose the contrary and derive a contradiction. Let then p be a condition, and f a term such that $p \Vdash f$ *is a function and domain* (f) *is an ordinal less than* $\boldsymbol{\kappa}$ *and* $(\forall x)\ (x \in \textit{domain}\,(f) \to f(x) \in J)$ *and* $\bigcup \textit{range}\,(f) = \boldsymbol{\kappa}$. By extending p, if necessary, we may suppose $p \Vdash$ *domain* $(f) = \boldsymbol{\lambda}$, for some ordinal $\lambda < \kappa$.

We let T_α be the set of "possible" members of $f(\alpha)$, given p. That is

$$T_\alpha = \{\beta < \kappa : (\exists q \succcurlyeq p)(q \Vdash \boldsymbol{\beta} \in f(\boldsymbol{\alpha}))\} \qquad (\alpha < \lambda)$$

Suppose that for some $\alpha < \lambda$ $T_\alpha \notin I$ then, by the discussion of Case 2 above, there is a $q \succcurlyeq p$ such that $W = \{\beta < \kappa : q \Vdash \boldsymbol{\beta} \in f(\boldsymbol{\alpha})\} \notin I$ and hence $q \Vdash f(\boldsymbol{\alpha}) \notin J$, which contradicts $p \Vdash f(\boldsymbol{\alpha}) \in J$. Thus we have $T_\alpha \in I$ for every $\alpha < \lambda$, and since I is κ-complete also

(7) $$\textstyle\bigcup_{\alpha<\lambda} T_\alpha \in I.$$

We show this contradicts

(8) $$p \Vdash \bigcup \textit{range}\,(f) = \boldsymbol{\kappa}.$$

Pick $\gamma \in (\kappa - \bigcup_{\alpha<\lambda} T_\alpha)$. (Possible, since $\kappa \notin I$.) For some $q \succcurlyeq p$, and some $\alpha < \lambda$, we have (by (8)) $q \Vdash \boldsymbol{\gamma} \in f(\boldsymbol{\alpha})$. Thus $\gamma \in T_\alpha$, which gives a contradiction.

Theorem 10 is no longer true if the cardinality of C is $\geqq \kappa$. (In fact, following Cohen [1], we can choose a set of conditions of power κ, so that $2^{\aleph_0} \geqq \boldsymbol{\kappa}$ in the Cohen extension. Measurable cardinals are strongly inaccessible so $\boldsymbol{\kappa}$ is not measurable in the extension.)

We now prove a strengthened version of Lemma 9.

LEMMA 11. *Suppose that κ is a cardinal number and that the class of conditions, C, has cardinality less than κ. It is true in the extension that if J is a $\boldsymbol{\kappa}$-complete non-principal prime ideal on the power set of $\boldsymbol{\kappa}$, then J is generated by some standard set I such that I is a non-principal κ-complete prime ideal in $S(\kappa)$ in the ground model.*

Proof. Let p be a condition and J a term such that $p \Vdash J$ *is a* $\boldsymbol{\kappa}$-*complete non-principal prime ideal in* $S(\boldsymbol{\kappa})$.

We must produce $q \succcurlyeq p$, and an ideal I such that 1) I is a $\boldsymbol{\kappa}$-complete non-principal prime ideal in $S(\kappa)$, 2) $q \Vdash I$ *generates J in* $S(\boldsymbol{\kappa})$.

The key fact is the following:
(*) there is a $q \geqslant p$ such that for every subset a of κ,

$$q \Vdash \mathbf{a} \in J \text{ or } q \Vdash \boldsymbol{\kappa} - \mathbf{a} \in J.$$

Granted this fact, it is routine to check that

$$I = \{a \mid q \Vdash \mathbf{a} \in J\}$$

is a κ-complete non-principal prime ideal such that

$$q \Vdash \boldsymbol{I} \subseteq J. \tag{9}$$

By Theorem 10, $\boldsymbol{I}$ generates some κ-complete prime ideal. By (9), we see that q forces this ideal to be J.

It remains to prove (*). Suppose, to the contrary, that for each $q \geqslant p$, there is a decomposition

$$\kappa = a_q^0 \bigcup a_q^1; \quad a_q^0 \bigcap a_q^1 = 0$$

such that neither $q \Vdash \mathbf{a}_q^0 \in J$ nor $q \Vdash \mathbf{a}_q^1 \in J$.

We divide κ into equivalence classes by putting $\alpha \sim \beta$ if for all $q \geqslant p$

$$\alpha \in a_q^0 \equiv \beta \in a_q^0.$$

Then there are at most $2^{card(C)}$ equivalence classes. Since $\boldsymbol{\kappa}$ is strongly inaccessible, so is κ, by (k), and therefore $2^{card(C)} = \lambda < \kappa$. Let $f : \lambda \to S(\kappa)$ enumerate these equivalence classes. By k) of §2, we see that $\boldsymbol{f} : \boldsymbol{\lambda} \to S(\boldsymbol{\kappa})$, and $\bigcup \text{range}(\boldsymbol{f}) = \boldsymbol{\kappa}$ in the extension. Since $p \Vdash J$ *is a* $\boldsymbol{\kappa}$*-complete prime ideal,* there exists $q \geqslant p$, and an ordinal $\alpha < \lambda$ such that

$$q \Vdash \boldsymbol{f}(\boldsymbol{\alpha}) \notin J.$$

Now by construction, for some $i \in \{0, 1\}$, we have $f(\alpha) \cap a_q^i = 0$. Thus $q \Vdash \boldsymbol{a}_q^i \bigcap \boldsymbol{f}(\mathbf{a}) = \mathbf{0}$. It follows that $q \Vdash \mathbf{a}_q^i \in J$ which gives the desired contradiction The proof of Lemma 11 is complete.

Remark. Using Lemma 11 and McAloon's result that strongly compact cardinals are preserved under mild Cohen extensions, Prikry (unpublished) has proved the following theorem:

If one can prove in ZF that every strongly compact cardinal is greater than some measurable cardinal, then one can also prove in ZF that every strongly compact cardinal λ is the λ-th measurable cardinal.

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